COVARIANTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS AND APPLICATIONS TO THE THEORY OF RULED SURFACES*

BY

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In the following paper, the author wishes to present the theory of the covariants of a system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0\,, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0\,, \end{aligned}$$

with the independent variable x. Some of these covariants present themselves, in a very elegant and simple form, in connection with the theory of the system adjoined to (1). The theory of covariants is here carried out to the same degree of completeness as has been previously done for the invariants. Then follows their geometrical interpretation.†

§ 1. *Fundamental properties of covariants.*

Any function of $y, z, y', z', \cdots, p_{ik}, p'_{ik}, \cdots, q_{ik}, q'_{ik}, \cdots$ which has the same value for system (1) and for any system equivalent to (1) by a transformation of the infinite group G,

$$(2)\quad \xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \alpha\delta - \beta\gamma \neq 0,$$

shall be called an *absolute covariant.* In particular if it does not contain y, z, y', z', etc., it is called an *invariant.* The invariants of system (1) have already been determined.

Let us assign weights zero to y and z, unity to p_{ik} and two to q_{ik}. Moreover let every differentiation increase the weight by one, and let the weight of a

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† To facilitate references to my previous papers on this subject, I shall quote them in the text by the initial words of their titles, viz.: Invariants, Geometry, Reciprocal Systems They have all been published in these Transactions: *Invariants*, vol. 2, p. 1 ; *Geometry*, vol. 2, p. 343 ; *Reciprocal Systems*, vol. 3, p. 60.

product be the sum of the weights of its factors. These conventions are somewhat different from those adopted in my paper on invariants, where I have used negative weights, following the example set by FORSYTH in a similar theory.* However the assignment of weights here made, which is essentially the same as that used by BOUTON,† appears after all to be more convenient.

In the proofs of the following theorems, I have utilized for my present purpose some ideas from the algebraic theory of quantics, as well as from the theory of invariants and covariants of a single linear differential equation.

We have already defined the term : absolute covariant. The function

$$C(y, z, y', z', \cdots, p_{ik}, p'_{ik}, \cdots, q_{ik}, q'_{ik}, \cdots)$$

is a *relative covariant*, if the equation

$$C = 0,$$

has as its consequence

$$\Gamma = 0,$$

where Γ denotes the same function of the transformed quantities

$$\eta, \zeta, \cdots, \pi_{ik}, \cdots, \rho_{ik}, \cdots,$$

as C of the corresponding original quantities.

Let us make the transformation

$$\xi = x, \qquad \eta = Cy, \qquad \zeta = Cz,$$

where C is a constant, and which is contained in the group G. Always denoting the transformed quantities by Greek letters, we shall then have

$$\eta^{(\lambda)} = Cy^{(\lambda)}, \qquad \zeta^{(\lambda)} = Cz^{(\lambda)}, \qquad \pi_{ik} = p_{ik}, \qquad \rho_{ik} = q_{ik}.$$

Therefore, any relative covariant must be homogeneous in y, z, y', z', etc. Any absolute covariant must be homogeneous in these quantities of degree zero.

Again, denoting by C a constant, let us make the transformation

$$\xi = Cx, \qquad \eta = y, \qquad \zeta = z,$$

which is also included in the group G. This gives

$$\eta^{(\lambda)} = C^{-\lambda} y^{(\lambda)}, \qquad \zeta^{(\lambda)} = C^{-\lambda} z^{(\lambda)}, \qquad \pi_{ik} = C^{-1} p_{ik}, \qquad \rho_{ik} = C^{-2} q_{ik},$$

which shows that every term of weight w is multiplied by C^{-w}.

A covariant must then be an isobaric function of the quantities upon which it depends, and of weight zero if it is an absolute covariant.

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* A. R. FORSYTH, Philosophical Transactions, vol. 179.

† C. L. BOUTON, American Journal of Mathematics, vol. 21 (1899).

Let $C_{\lambda, w}$ be an integral rational function of y, z, p_{ik}, q_{ik} and of the derivatives of these quantities, which is a covariant. Then it is necessarily homogeneous, say of degree λ in y, z, y', z', etc., and isobaric, say of weight w. Moreover, let $C_{\lambda, w}$ be irreducible in the sense that it is not possible to factor it into rational factors of lower weight and degree. Let us consider the effect upon $C_{\lambda, w}$ of a transformation of the dependent variables y and z alone.

Let the transformation be given in the form

$$(3) \qquad y = \alpha_{11}\eta + \alpha_{12}\zeta, \qquad z = \alpha_{21}\eta + \alpha_{22}\zeta,$$

or

$$(4) \qquad \Delta\eta = \alpha_{22}y - \alpha_{12}z, \qquad \Delta\zeta = -\alpha_{21}y + \alpha_{11}z,$$

$$\Delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21},$$

where α_{ik} are arbitrary functions of x.

If the coefficients of the transformed system of differential equations are again denoted by π_{ik} and ρ_{ik}, we shall have (*Invariants*, p. 3, if there we put $n = m = 2$),

$$\Delta\pi_{11} = 2(\alpha'_{11}\alpha_{22} - \alpha'_{21}\alpha_{12}) + p_{11}\alpha_{11}\alpha_{22} + p_{12}\alpha_{21}\alpha_{22} - p_{21}\alpha_{11}\alpha_{12} - p_{22}\alpha_{21}\alpha_{22},$$

. . . .

$$(5) \quad \Delta\rho_{11} = \alpha''_{11}\alpha_{22} - \alpha''_{21}\alpha_{12} + p_{11}\alpha'_{11}\alpha_{22} + p_{12}\alpha'_{21}\alpha_{22} - p_{21}\alpha'_{11}\alpha_{12} - p_{22}\alpha'_{21}\alpha_{22}$$
$$+ q_{11}\alpha_{11}\alpha_{22} + q_{12}\alpha_{21}\alpha_{22} - p_{21}\alpha_{11}\alpha_{12} - p_{22}\alpha_{21}\alpha_{22},$$

. . . .

where only one equation of each kind has been written down.

Let $\Gamma_{\lambda, w}$ denote what $C_{\lambda, w}$ becomes when y, z, p_{ik}, etc., are replaced by η, ζ, π_{ik}, etc. Since $C_{\lambda, w}$ is a relative covariant, the equation $\Gamma_{\lambda, w} = 0$, must be a consequence of $C_{\lambda, w} = 0$. But equations (4), (5) and those deducible from these by differentiation, show clearly that in place of every term of weight w in C, we shall have in Γ a collection of terms of weight w plus terms of lower weight. But these latter terms must annihilate each other if $C_{\lambda, w}$ is a covariant, i. e., their sum must be identically zero. For they cannot vanish in consequence of $C_{\lambda, w} = 0$, since their aggregate is rational and of lower weight than w, while $C_{\lambda, w}$ cannot be factored into rational factors of lower weight. Now it is clear from equations (4) and (5) that the terms of weight w in $\Gamma_{\lambda, w}$ when expressed in terms of y, z, p_{ik}, etc., will contain only the quantities α_{ik} themselves and not their derivatives. We see then that there must be an equation of the form

$$(6) \qquad \Gamma_{\lambda, w} = f(\alpha_{ik})C_{\lambda, w},$$

where $f(\alpha_{ik})$ depends only upon the four arguments indicated.

Moreover, from (4) and (5) we notice that π_{ik} and ρ_{ik} are homogeneous functions of degree zero, and that η and ζ are homogeneous functions of degree -1 of the quantities α_{ik}, so that $\Gamma_{\lambda, w}$, and therefore $f(\alpha_{ik})$, will be a homogeneous function of degree $-\lambda$ of its arguments. Further it is clear, from (4) and (5), that $f(\alpha_{ik})$ can be written in the form

$$f(\alpha_{ik}) = \frac{\phi(\alpha_{ik})}{\Delta^{\mu}}, \tag{7}$$

where $\phi(\alpha_{ik})$ is an integral rational function of its arguments of degree $-\lambda + 2\mu$, since $f(\alpha_{ik})$ is of degree $-\lambda$. Therefore

$$\Gamma_{\lambda, w} = \frac{\phi(\alpha_{ik})}{\Delta^{\mu}} C_{\lambda, w}. \tag{8}$$

Just in the same way if we replace in $C_{\lambda, w}$ the quantities y, z, p_{ik}, etc., by their values in terms of η, ζ, π_{ik}, etc., obtained by solving (4) and (5), we should find

$$C_{\lambda, w} = \frac{\phi(A_{ik})}{\Delta'^{\mu}} \Gamma_{\lambda, w}, \tag{9}$$

where A_{ik} denotes the minor of α_{ik} in Δ divided by Δ, and where

$$\Delta' = A_{11}A_{22} - A_{12}A_{21} = \frac{1}{\Delta}. \tag{10}$$

From (8) and (9) we have by multiplication

$$\phi(\alpha_{ik})\phi(A_{ik}) = 1,$$

or writing out the four arguments

$$\phi(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})\phi\left(\frac{\alpha_{22}}{\Delta}, -\frac{\alpha_{12}}{\Delta}, -\frac{\alpha_{21}}{\Delta}, \frac{\alpha_{11}}{\Delta}\right) = 1,$$

or, since ϕ was homogeneous of degree $2\mu - \lambda$,

$$\phi(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})\phi(\alpha_{22}, -\alpha_{12}, -\alpha_{21}, \alpha_{11}) = \Delta^{2\mu-\lambda}.$$

But Δ cannot be factored into two integral rational factors,* so that the factors on the left member of this equation must themselves be powers of Δ. Therefore $f(\alpha_{ik})$ is a power of Δ. Moreover since $f(\alpha_{ik})$ is of degree $-\lambda$ while Δ is of the second degree, we shall have

$$\Gamma_{\lambda, w} = \Delta^{-\frac{\lambda}{2}} C_{\lambda, w},$$

* E. B. Elliott, *Algebra of Quantics*, § 14.

for it becomes clear that a numerical factor k different from unity is inadmissible, by applying the identical transformation.

For our proof it was convenient to take the transformation in the form (3), solved for y and z. If we write instead the transformation in the form

$$\eta = \alpha y + \beta z, \qquad \zeta = \gamma y + \delta z,$$

we know now that a rational covariant $C_{\lambda, w}$ of weight w and of degree λ is transformed in accordance with the equation

$$\Gamma_{\lambda\ w} = (\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}\, C_{\lambda, w}.$$

Moreover, as the right member must obviously be rational in α, β, γ, δ, we get this theorem:

There are no rational covariants of odd degree for a binary system of linear homogeneous differential equations.

It is obvious how this theorem will generalize for m-ary systems.

Let us now make a transformation of the independent variable

$$\xi = \xi(x).$$

We shall find, by merely consulting the formulæ [*Invariants*, equations (56)], that every term of weight w in $\Gamma_{\lambda, w}$ is equal to a corresponding term of $C_{\lambda, w}$ multiplied by

$$(\xi')^{-w},$$

plus terms of lower weight. But the aggregate of the latter terms must vanish identically, since it cannot vanish as a consequence of the equation $C_{\lambda, w} = 0$, which is isobaric of weight w. Therefore we shall have

$$\Gamma_{\lambda, w} = \frac{1}{(\xi')^{w}}\, C_{\lambda, w}.$$

Combining our results we have the following fundamental theorem:

If $C_{\lambda, w}$ is an integral rational covariant of degree λ and of weight w, it is transformed by the transformation

$$\xi = \xi(x), \qquad \eta = \alpha(x) y + \beta(x) z, \qquad \zeta = \gamma(x) y + \delta(x) z,$$

in accordance with the equation

$$\Gamma_{\lambda, w} = \frac{(\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}}{(\xi')^{w}}\, C_{\lambda, w}. \tag{11}$$

Moreover its degree λ is necessarily even.

The product $C^{m_1}_{2\lambda_1, w_1} C^{m_2}_{2\lambda_2, w_2}$ is an absolute rational covariant, if and only if two integers m_1 and m_2, positive or negative, can be found satisfying the conditions

$$m_1 \lambda_1 + m_2 \lambda_2 = 0,$$
$$m_1 w_1 + m_2 w_2 = 0,$$

i. e., since the possibility $m_1 = m_2 = 0$ must be excluded, *an absolute covariant can be formed from* $C_{2\lambda_1, w_1}$ *and* $C_{2\lambda_2, w_2}$, *if and only if*

$$\lambda_1 w_2 - \lambda_2 w_1 = 0.$$

It follows at once that if two covariants of the same degree can be combined into an absolute covariant, their weights must be equal, and conversely.

§ 2. *Construction of covariants and semi-covariants.*

If we denote the left members of (1) by Y and Z, it is clear from our definitions of the transformed coefficients, that corresponding to the transformations

$$x = f(\xi), \qquad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

Y and Z are transformed in accordance with the equations

$$(12) \qquad Y = \frac{\alpha \bar{Y} + \beta \bar{Z}}{(\xi')^2}, \qquad Z = \frac{\gamma \bar{Y} + \delta \bar{Z}}{(\xi')^2},$$

where $\bar{Y}$ and $\bar{Z}$ denote the left members of the transformed system of differential equations.

Since moreover the solutions U, V of the system of differential equations adjoined to (1) are transformed cogrediently with y and z (*Reciprocal Systems*, § 2), the equations (12) will also apply to the left members of the adjoined system

$$(13) \qquad \begin{aligned} &U'' + p_{11} U' + p_{12} V' + \{ q_{11} + \tfrac{1}{4}(u_{11} - u_{22}) \} U + (q_{12} + \tfrac{1}{2} u_{12}) V = 0, \\ &V'' + p_{21} U' + p_{22} V' + (q_{21} + \tfrac{1}{2} u_{21}) U + \{ q_{22} + \tfrac{1}{4}(u_{22} - u_{11}) \} V = 0. \end{aligned}$$

Moreover this will still be true if in (13), we write y and z in place of U and V on account of the cogrediency of these quantities.

We have then at once two covariants of degree and weight 2. They are

$$(14) \qquad \begin{aligned} C_1 &= \begin{vmatrix} y'' + p_{11} y' + p_{12} z' + q_{11} y + q_{12} z, & y \\ z'' + p_{21} y' + p_{22} z' + q_{21} y + q_{22} z, & z \end{vmatrix}, \\ C_2 &= \begin{vmatrix} y'' + p_{11} y' + p_{12} z' + \{ q_{11} + \tfrac{1}{4}(u_{11} - u_{22})\} y + (q_{12} + \tfrac{1}{2} u_{12}) z, & y \\ z'' + p_{21} y' + p_{22} z' + (q_{21} + \tfrac{1}{2} u_{21}) y + \{ q_{22} + \tfrac{1}{4}(u_{22} - u_{11})\} z, & z \end{vmatrix}. \end{aligned}$$

Of course C_1 vanishes for solutions of (1), and C_2 for (13). If we write

$$(15) \qquad C = 2(C_2 - C_1) = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz.$$

C is also a covariant of weight and degree 2. This may, of course, be tested analytically.

The covariant C vanishes identically, i. e., for arbitrary functions y and z of x, if and only if the integrating ruled surface of system (1) is of the second order. For the necessary and sufficient conditions for this are (*Geometry*, p. 360),

$$u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

We shall call a function of y, z, y', z', p_{ik}, q_{ik}, etc., a *semi-covariant* if it possesses the covariant property for transformations of the dependent variables y and z alone, whether or not it also has the covariant property for transformations of x.

Now v_{ik} and w_{ik} are cogredient with u_{ik} for transformations of the dependent variables (*Invariants*, pp. 8, 9, 10). Therefore

$$(16) \qquad \begin{aligned} E &= v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz, \\ F &= w_{12}z^2 - w_{21}y^2 + (w_{11} - w_{22})yz, \end{aligned}$$

are semi-covariants of weight three and four respectively, and both of the second degree.

If then we make the transformation

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad \Delta = \alpha\delta - \beta\gamma,$$

and denote the transformed functions by dashes, we shall have

$$C = \Delta\bar{C}, \qquad E = \Delta\bar{E}, \qquad F = \Delta\bar{F},$$

so that the Wronskians

$$(17) \qquad \begin{aligned} (C'E) &= C'E - CE', \qquad (E'F) = E'F - EF', \\ (F'C) &= F'C - FC', \end{aligned}$$

are also semi-covariants obviously of degree 4.

We have further

$$\frac{C'}{C} = \frac{\bar{C}'}{\bar{C}} + \frac{\Delta'}{\Delta},$$

and from the fundamental equations

$$p_{11} + p_{22} = \bar{p}_{11} + \bar{p}_{22} - 2\frac{\Delta'}{\Delta},$$

whence

$$2\frac{C'}{C} + p_{11} + p_{22} = 2\frac{\bar{C}'}{\bar{C}} + \bar{p}_{11} + \bar{p}_{22},$$

so that

(18) $$G = 2C' + (p_{11} + p_{22})C$$

is a new semi-covariant, and obviously in the same way

(19) $$H = 2E' + (p_{11} + p_{22})E, \qquad M = 2F' + (p_{11} + p_{22})F.$$

It is obviously sufficient to consider only those semi-covariants and covariants involving no higher derivatives of y and z than the first. For if a covariant contains higher derivatives of y and z, we can express them in terms of y, z, y', z' by means of equations (1) and others deduced therefrom by differentiation.

So as to proceed in an orderly manner, let us first determine all independent semi-covariants containing besides y, z, y', z', merely the quantities p_{ik}, p'_{ik} and q_{ik}. We have already found one such, namely C. We can find another by forming $G - E$. If we make use of the definitions (*Invariants*, equations (32)), of the quantities v_{ik}, we shall have

(20) $$\begin{aligned} N = G - E = \{2p_{22}u_{12} + p_{12}(u_{11} - u_{22})\}z^2 - \{2p_{11}u_{21} - p_{21}(u_{11} - u_{22})\}y^2 \\ + \{2p_{21}u_{12} - 2p_{12}u_{21} + (p_{11} + p_{22})(u_{11} - u_{22})\}yz \\ + 4u_{12}zz' - 4u_{21}yy' + 2(u_{11} - u_{22})(yz' + y'z), \end{aligned}$$

a semi-covariant of degree 2 and weight 3.

If we examine the system of partial differential equations, of which the seminvariants and semi-covariants involving only the above variables are the solutions, we shall find that it consists of 12 independent equations with 16 independent variables. It has therefore four independent solutions. There must then be four seminvariants and absolute semi-covariants depending upon these 16 variables. Of these we know the seminvariants I and J, and the two relative semi-covariants C and N. There must be one more, which could be found by integrating the system of partial differential equations just mentioned. It can also be found in another manner. Put

(21) $$\rho = 2y' + p_{11}y + p_{12}z, \qquad \sigma = 2z' + p_{21}y + p_{22}z.$$

Then it may be verified that, for transformations of the dependent variables, ρ and σ are cogredient with y and z. Therefore

(22) $$P = z\rho - y\sigma = 2(y'z - yz') + p_{12}z^2 - p_{21}y^2 + (p_{11} - p_{22})yz$$

is a semi-covariant of degree 2 and of weight 1. Moreover it is clear that C, P and N are independent of each other and of I and J, so that all semi-covariants involving only y, z, y', z', p_{ik}, p'_{ik}, q_{ik} have been found. N can be written more simply,

$$(23) \qquad N = 2(u_{12}z\sigma - u_{21}y\rho) + (u_{11} - u_{22})(z\rho + y\sigma).$$

If we wish to find all seminvariants and semi-covariants involving p''_{ik} and q'_{ik} besides the former variables, we set up the system of partial differential equations satisfied by them. It contains 24 independent variables and 16 independent equations. Therefore, there must be 8 such seminvariants and semi-covariants. But we know them already. They are

$$(24) \qquad I,\ I',\ J,\ J',\ K; \quad \frac{P}{C},\ \frac{N}{C},\ \frac{E}{C};$$

for these are independent. That this is so becomes clear from the following argument. If there exists a relation between them E/C must certainly occur in it. For, suppose it did not. The relation could not also be free of all three of the quantities I', J', K, as we have already seen that I, J, P/C and N/C are independent. If it contains one of them, say K, let us solve for K and find

$$K = f\left(I,\ J,\ I',\ J',\ \frac{P}{C},\ \frac{N}{C}\right).$$

But this is manifestly impossible if either or both of the last two arguments actually appear. For K does not contain y, z, y', z' while P/C and N/C are independent functions of these variables. But this equation is impossible even if P/C and N/C do not appear, since I, I', J, J', K are independent seminvariants, as I have shown on a former occasion.

If there is a syzygy between the quantities (24) it must, therefore, contain E/C. Suppose that it be solved for E/C. Then we should have

$$\frac{E}{C} = f\left(I,\ I',\ J,\ J',\ K,\ \frac{P}{C},\ \frac{N}{C}\right).$$

But again the last two arguments cannot appear. For P and N are independent functions of y' and z', which are not contained in E/C at all. But even if they are suppressed, such an equation is impossible, for it would make the semi-covariant E/C equal to a seminvariant containing no y or z. Thus we have shown that the quantities (24) are independent.

Again making use of a system of partial differential equations, we notice that there are $32 - 20 = 12$ independent seminvariants and absolute semi-covariants involving the additional variables $p_{ik}^{(3)}$ and q''_{ik}. They are

$$(25) \qquad I, I', I'', J, J', J'', K, K', L, \frac{P}{C}, \frac{N}{C}, \frac{E}{C},$$

for these, it is easily seen, are independent. F/C, which obviously belongs to this same class of semi-covariants must then be expressible in terms of the other twelve. Moreover the syzygy which F/C and these quantities (25) verify must be independent of P/C and N/C since F/C does not depend upon y' or z'. It may be put into the form

$$f(C, E, F;\ I, I', I'', J, J', J'', K, K', L) = 0,$$

where f is homogeneous in C, E and F. We need not at present investigate this syzygy any farther.

No new semi-covariants will appear if we continue our search, and as we have seen in a former paper, all new seminvariants which appear are derivatives of I, J, K and L. (*Invariants*, p. 10.)

Let us put

$$(26) \qquad \frac{C}{P} = \gamma, \qquad \frac{E}{P} = \epsilon, \qquad \frac{N}{P} = \nu.$$

Then any absolute covariant must be a function of the seminvariants I, J, K, L, of their derivatives and of γ, ϵ and ν. If we consider only such covariants, containing no higher derivatives of p_{ik} and q_{ik} then $p_{ik}^{(3)}$ and $q_{ik}^{(2)}$ respectively, we shall find them as solutions of the system of partial differential equations

$$(27) \qquad \begin{aligned} &Y_1 f + \gamma\frac{\partial f}{\partial\gamma} + 2\epsilon\frac{\partial f}{\partial\epsilon} + 2\nu\frac{\partial f}{\partial\nu} = 0, \\ &Y_2 f - 4\gamma\frac{\partial f}{\partial\epsilon} + 2\gamma\frac{\partial f}{\partial\nu} = 0, \\ &Y_3 f = Y_4 f = Y_5 f = 0, \end{aligned}$$

where $Y_1 f \cdots Y_5 f$ are the operators defined in a former paper, being the left members of the system of partial differential equations satisfied by the invariants. [*Invariants*, equations (77).]

The equations (27) are independent and contain twelve independent variables. Therefore, there must be $12 - 5 = 7$ independent absolute invariants and covariants, or eight independent relative invariants and covariants.

Of these we know five to be invariants, viz.: θ_4, θ_6, θ_{10}, θ_{15} and θ_{18}. *The three covariants are*

$$(28) \qquad C_1 = P, \qquad C_2 = C, \qquad C_3 = E + 2N,$$

where, as in the case of invariants, the subscript denotes the weight. The three covariants are all of degree two. *All other covariants can be expressed in terms of* C_1, C_2, C_3 *and invariants.* As for the invariants there are an infinite number of independent ones, but they are all functions of the seminvariants I, J, K, L and of their derivatives.

As absolute covariants, we may take

$$(29) \qquad \frac{C_2^4}{\theta_4 C_1^4} \quad \text{and} \quad \frac{C_3^2}{\theta_4 C_1^2}.$$

for it is clear that the covariants (28) alone cannot be combined to give both absolute covariants, since they are all of the same degree but of different weights.

§ 3. *Geometrical interpretation of* ρ, σ *and* P.

If (y_k) and (z_k), for $(k = 1, 2, 3, 4)$, are the members of a simultaneous fundamental system of solutions of (1), we interpret y_k and z_k as the homogeneous coördinates of two points P_y and P_z. As x changes, P_y and P_z describe two curves C_y and C_z, and the line joining them L_{yz} generates the integrating ruled surface S of (1), which remains invariant under all transformations of the group G. (*Geometry*, § 2.)

If in (21) we put $y = y_k$. $z = z_k$, we obtain

$$(30) \qquad \begin{aligned} \rho_k &= 2y_k' + p_{11}y_k + p_{12}z_k, \\ \sigma_k &= 2z_k' + p_{21}y_k + p_{22}z_k, \end{aligned} \qquad (k=1, 2, 3, 4),$$

which quantities we may again interpret as the homogeneous coördinates of two points P_ρ and P_σ. Clearly, P_ρ is a point of the plane tangent to the integrating ruled surface S at P_y, and P_σ is a point of the plane tangent to S at P_z.

If the points P_y and P_z be transformed into two other points and $P_{\bar{y}}$ $P_{\bar{z}}$ of the line L_{yz} by the equations

$$(31) \qquad y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z},$$

then, as shown in § 2, P_ρ and P_σ will be transformed cogrediently into $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$, where

$$(32) \qquad \rho = \alpha\bar{\rho} + \beta\bar{\sigma}, \qquad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma}.$$

Therefore, if system (1) be transformed into an equivalent system by transformation (31), and if the corresponding quantities $\bar{\rho}$ and $\bar{\sigma}$ be formed for it, the points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ will lie on the line $L_{\rho\sigma}$ joining P_ρ and P_σ just as the points $P_{\bar{y}}$ and $P_{\bar{z}}$ lie on the line L_{yz} joining P_y and P_z.

Thus, we have, by equations (30) *a straight line* $L_{\rho\sigma}$ *corresponding to every generator* L_{yz} *of* S. *Moreover there is a one-to-one correspondence between the points of these two lines, which we now propose to investigate. Of course the lines* $L_{\rho\sigma}$ *generate a second ruled surface, which we will denote by* S'.

If $p_{12} = p_{21} = 0$, the curves C_y and C_z described on S by P_y and P_z are asymptotic lines. (*Reciprocal Systems*, p. 69.) Equations (30) show that then the points P_ρ, P_σ are points of the tangents to these curves at P_y and P_z respectively. But as we can always reduce a system of form (1) to a form (for instance the semi-canonical form), for which $p_{12} = p_{21} = 0$, we see that P_ρ and P_σ are points on the tangents to the asymptotic lines which pass through P_y and P_z respectively.

Now, through each point of the generator L_{yz}, or g, of the surface S passes an asymptotic line. The tangents of all these asymptotic lines along g are the generators of the second kind of an hyperboloid, three of whose generators of the first kind are g, g' and g'', where g' and g'' are two generators of the surface S infinitesimally close to g. Let us call this hyperboloid H, *the hyperboloid osculating* S *along* g.

We have seen that, according to (30), we have a point P' corresponding to every point P on g. Moreover P' is a point on the tangent to the asymptotic line passing through P, and further the locus of all points P', corresponding to all of the points P of g, is a straight line $L_{\rho\sigma}$.

This line $L_{\rho\sigma}$ *must then be a generator of the same kind as* g *on the hyperboloid* H, *osculating the surface* S *at* g.

Omitting the subscripts, and differentiating (30), we find

$$\rho' = 2y'' + p_{11}y' + p_{12}z' + p'_{11}y + p'_{12}z,$$
$$\sigma' = 2z'' + p_{21}y' + p_{22}z' + p'_{21}y + p'_{22}z,$$

whence, making use of (1),

$$\rho' = -p_{11}y' - p_{12}z' + (p'_{11} - 2q_{11})y + (p'_{12} - 2q_{12})z,$$
$$\sigma' = -p_{21}y' - p_{22}z' + (p'_{21} - 2q_{21})y + (p'_{22} - 2q_{22})z.$$

If the values of y' and z' in terms of y, z, ρ, σ be substituted from (30), the following equations will be obtained

$$\text{(33)}\qquad \begin{aligned} R &= 2\rho' + p_{11}\rho + p_{12}\sigma = u_{11}y + u_{12}z, \\ S &= 2\sigma' + p_{21}\rho + p_{22}\sigma = u_{21}y + u_{22}z \end{aligned}$$

where R and S are merely abbreviations for the left members, and where the quantities u_{ik} are the same as those in my paper on *Invariants*, equations (20).

Equations (33) show that if the planes tangent to the ruled surface S', of which $L_{\rho\sigma}$ is a generator, are constructed at P_ρ and P_σ, they will intersect the line L_{yz} in the points $(u_{11}y_k + u_{12}z_k)$ and $(u_{21}y_k + u_{22}z_k)$ respectively. Or, in other words, the lines joining ρ_k with $u_{11}y_k + u_{12}z_k$, and σ_k with $u_{21}y_k + u_{22}z_k$ are tangents of the ruled surface S' at P_ρ and P_σ respectively.

In particular then, if $u_{12} = u_{21} = 0$, *the lines which are tangent to the asymptotic curves of the surface* S *at* P_y *and* P_z *are also tangent to the ruled surface* S'. But we can find a simpler and more fundamental interpretation of the conditions $u_{12} = u_{21} = 0$.

Consider three consecutive generators g_{-1}, g_0, g_1 of the ruled surface S. The hyperboloid H_0 osculating S along g_0 is determined by these three lines. On H_0 we have a line $L_{\rho\sigma}$, or for short h_0, which is the generator of S' corresponding to the generator g_0 of S. Consider a fourth generator g_2 of S consecutive to g_1. The lines g_0, g_1, g_2 determine the hyperboloid H_1 osculating S along g_1. There is upon it a line h_1 which is the next generator of S'. Any tangent to S' along h_0 must intersect h_0 and h_1. If it is at the same time tangent to an asymptotic line of S at any point of g_0 it must intersect the lines g_{-1}, g_0, g_1 also. Such a line must then intersect the five lines g_{-1}, g_0, g_1, h_0, h_1. But since h_0 is on the hyperboloid determined by g_{-1}, g_0, g_1 we may suppress h_0, since any line intersecting the first three will also intersect h_0. Therefore we can say that such a line must intersect the four lines g_{-1}, g_0, g_1, h_1. But any line intersecting g_{-1}, g_0, g_1 is a generator of the second kind of the hyperboloid H_0, and any line intersecting g_0, g_1, h_1 is a generator of the second kind of the hyperboloid H_1, since g_0, g_1, h_1 are generators of the first kind on H_1.

Therefore any line, which is tangent to an asymptotic curve of S *at a point of* g, *and which is at the same time tangent to the surface* S' *at a point of the generator of that surface corresponding to* g, *is common to two consecutive osculating hyperboloids of the surface* S. *Or, in other words, such a line intersects four consecutive generators of the surface* S.

There are, in general, two such lines, since four lines in space have two real, imaginary or coincident straight line intersectors.

Now we shall see in the next paragraph, that there exists, in general, two and only two points on each generator, such that if their loci be taken as fundamental curves C_y and C_z, the corresponding system of differential equations, will satisfy the conditions $u_{12} = u_{21} = 0$. *They are, therefore, the points at which tangents can be drawn to the surface, which have four consecutive points in common with it.*

CAYLEY * has called such a point of a surface, at which a four-point tangent can be drawn, *a flecnode*, the tangent itself the *flecnode tangent*, and the locus of all of the flecnodes, the *flecnode curve.* It should be noticed that the flecnode tangents are not in general tangents to the flecnode curve. They are however tangent to the asymptotic curve, which passes through the flecnode.

The flecnode curve intersects every generator of the surface twice, and if its two branches be taken as fundamental curves, the conditions $u_{12} = u_{21} = 0$ *will be satisfied for the corresponding system of differential equations.*

The problem of effecting such a transformation of (1) as to make $u_{12} = u_{21} = 0$, i. e., the problem of determining the flecnode curve will be solved in the next paragraph.

We have seen that $L_{\rho\sigma}$ is a line on the hyperboloid H osculating S along L_{yz}. The question naturally arises whether this generator of H is essentially distinguished geometrically from every other. The question is to be answered in the negative. For, if in the covariant P, we replace ρ and σ by $\bar{\rho} = \rho + \lambda y$ and $\bar{\sigma} = \sigma + \lambda z$ respectively, where λ is an arbitrary seminvariant function of x, the covariant is not changed, and $\bar{\rho}$, $\bar{\sigma}$ are, for transformations of the dependent variables, cogredient with y and z as well as ρ and σ. Moreover a change of the independent variable x transforms ρ and σ into $\bar{\rho}$ and $\bar{\sigma}$. But the line joining the points $\bar{\rho}_k$ and $\bar{\sigma}_k$ is an arbitrary generator of H belonging to the same set as L_{yz}. Therefore, it is properly this set of generators on H, rather than any one of them, which is covariantly connected with the surface.

Still the line $L_{\rho\sigma}$ is distinguished among these generators in a pseudo-geometrical manner. Suppose $p_{ik} = 0$, so that the curves C_y and C_z are asymptotic lines, the system (1) having been reduced to the semi-canonical form. Then

$$\rho = 2y', \qquad \sigma = 2z'.$$

Consider the three consecutive generators g_{-1}, g_0, g_{+1} of S as belonging to the values of x

$$x_0 - \Delta x, \qquad x_0, \qquad x_0 + \Delta x,$$

respectively, where Δx is an infinitesimal. Construct the tangent to C_y at P_y. It meets the three generators g_{-1}, g_0, g_{+1} since C_y is an asymptotic curve. The coördinates of the three points of intersection, A, B and C, are

$$y_k - y'_k \Delta x, \qquad y_k, \qquad y_k + y'_k \Delta x,$$

so that the point P_ρ, whose coördinates are proportional to y'_k, is the harmonic conjugate of B with respect to A and C. Similarly along C_z. We may therefore imagine that the line $L_{\rho\sigma}$ is selected as follows: We consider three

* CAYLEY, *On the Singularities of Surfaces*, Mathematical Papers, vol. II, p. 29.

generators of the surface S corresponding to three values of x forming an arithmetical progression of common difference d. Upon the hyperboloid determined by these three lines we construct the generator which is the harmonic conjugate of the middle line with respect to the other two. As the common difference d approaches the limit zero the hyperboloid approaches as a limit the osculating hyperboloid and the fourth generator approaches as a limit the line $L_{\rho\sigma}$.

Consider the totality of osculating hyperboloids. It contains ∞^2 straight lines which therefore form a congruence. If we choose from the lines of this congruence a single infinity, one on each hyperboloid, we get a ruled surface covariant with S. The surface S' generated by $L_{\rho\sigma}$ is such a one. But obviously there are an infinite number of such surfaces. In particular we may take such lines of the congruence, one on each hyperboloid, which intersect a given line, and thus deduce from the given surface another belonging to a special linear complex. To accomplish this, we need merely set up the system of differential equations for

$$y + \lambda\rho, \qquad z + \lambda\sigma,$$

and determine λ so that this system may have an integrating ruled surface belonging to a special linear complex. This, it seems, should be of importance for the integration of a system of form (1). In fact, if λ has been so determined, the adjunction of λ to the domain of rationality, would reduce the transformation group (generalized in the sense of PICARD and VESSIOT), of system (1) to a subgroup leaving invariant the linear complex.

If system (1) already has a ruled surface belonging to a linear complex, we can thus deduce from it another belonging to a linear congruence, and if the ruled surface of (1) belongs to such a congruence in the first place, such a transformation will lead to a ruled surface of the second order.

We can obtain another congruence associated with the surface, and make similar remarks about it, in the following manner. The solutions U_k, V_k of the system adjoined to (1) were defined as the coördinates of the planes tangent to the surface S at P_y and P_z respectively. (*Reciprocal Systems*, §§ 1 and 2.) The relations (*Reciprocal Systems*, equations (27))

$$\sum y_k U_k = 0, \qquad \sum z_k U_k = 0, \qquad \sum y'_k U_k = 0,$$
$$\sum y_k V_k = 0, \qquad \sum z_k V_k = 0, \qquad \sum z'_k V_k = 0,$$

show this. But we may also interpret U_k and V_k as point coördinates. Then, these equations show that U_k and V_k are the poles of the planes tangent to S at P_y and P_z with respect to the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

We have again a projective correspondence between the points of the line L_{yz} and its polar L_{UV}. If we join corresponding points of the two lines we obtain a hyperboloid. There is one such for every generator of S, and clearly we have again a congruence, made up of the first set of generators of these hyperboloids, to which the surface S belongs.

It is not my present purpose to follow out these ideas any further. I shall, however, deduce the system of differential equations of which ρ_k and σ_k are the solutions, and whose integrating ruled surface is S'.

If we solve (33) for y and z, we obtain

$$(34)\qquad \begin{aligned} Jy &= u_{22} R - u_{12} S, \\ Jz &= -u_{21} R + u_{11} S, \end{aligned} \qquad J = u_{11} u_{22} - u_{12} u_{21},$$

If (33) be differentiated, and the values of y' and z' in terms of y, z, ρ, σ be substituted the following equations result:

$$\begin{aligned} 2R' &= u_{11}\rho + u_{12}\sigma + (2u'_{11} - u_{11}p_{11} - u_{12}p_{21})y + (2u'_{12} - u_{11}p_{12} - u_{12}p_{22})z, \\ 2S' &= u_{21}\rho + u_{22}\sigma + (2u'_{21} - u_{21}p_{11} - u_{22}p_{21})y + (2u'_{22} - u_{21}p_{12} - u_{22}p_{22})z. \end{aligned}$$

The quantities u'_{ik} may be expressed in terms of the quantities p_{ik} and v_{ik} (*Invariants*, equations (32)). If this be done, if, moreover, the equations be multiplied by J on both members, and if use be made of (34), the equations become

$$(35)\qquad \begin{aligned} 2JR' - Ju_{11}\rho - Ju_{12}\sigma + t_{11} R + t_{12} S = 0, \\ 2JS' - Ju_{21}\rho - Ju_{22}\sigma + t_{21} R + t_{22} S = 0, \end{aligned}$$

where we have put

$$(36)\qquad \begin{aligned} t_{11} &= Jp_{11} + u_{21}v_{12} - u_{22}v_{11}, \\ t_{12} &= Jp_{12} - u_{11}v_{12} + u_{12}v_{11}, \\ t_{21} &= Jp_{21} + u_{21}v_{22} - u_{22}v_{21}, \\ t_{22} &= Jp_{22} - u_{11}v_{22} + u_{12}v_{21}. \end{aligned}$$

Performing the differentiations indicated, inserting the values of R and S from (33), and collecting terms, we find the required system of differential equations:

$$(37)\qquad \begin{aligned} 4J\rho'' + 2(Jp_{11} + t_{11})\rho' + 2(Jp_{12} + t_{12})\sigma' + (2Jp'_{11} - Ju_{11} \\ + t_{11}p_{11} + t_{12}p_{21})\rho + (2Jp'_{12} - Ju_{12} + t_{11}p_{12} + t_{12}p_{22})\sigma = 0, \\ 4J\sigma'' + 2(Jp_{21} + t_{21})\rho' + 2(Jp_{22} + t_{22})\sigma' + (2Jp'_{21} - Ju_{21} \\ + t_{21}p_{11} + t_{22}p_{21})\rho + (2Jp'_{22} - Ju_{22} + t_{21}p_{12} + t_{22}p_{22})\sigma = 0. \end{aligned}$$

Suppose that system (1) has the semi-canonical form, so that $p_{ik} = 0$. In general (37) will not then also assume the semi-canonical form. It will, if and only if $t_{ik} = 0$ together with $p_{ik} = 0$. But then, as equations (36) show, either $J = 0$, or $v_{11} = v_{12} = v_{21} = v_{22} = 0$. But if J were zero, equations (33) show that the planes tangent to S' at P_ρ and P_σ would intersect L_{yz} in the same point. But this can happen only in one of two ways. Either these planes are identical, in which case the surface S' is developable, or else the generator $L_{\rho\sigma}$ of the surface S' intersects L_{yz}. This latter case is only possible for singular values of x for which the osculating hyperboloid degenerates. For else the surface S would be developable, which it can never be. (*Geometry*, p. 347.) The case that S' is developable must also be excluded for the same reason. For then ρ and σ could verify no system of differential equations of the form here considered.

In order that (1) and (37) may be simultaneously put into the semi-canonical form, it is therefore necessary to have

$$p_{ik} = v_{ik} = 0, \tag{38}$$

whence u_{ik} and therefore q_{ik} are seen to be constants. Conversely, if $p_{ik} = 0$ and q_{ik} are constants, we find $v_{ik} = t_{ik} = 0$, so that (1) and (37) will both have the semi-canonical form, but from (38) follows also $w_{ik} = 0$, where the quantities w_{ik} have been defined in my paper on Invariants, equations (39). But these conditions make all the minors of the second order in the determinant

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix}$$

vanish, *so that the surface S, and consequently also S', is contained in a linear congruence.* (*Geometry*, p. 356.)

Let us notice that in the case here considered, $p_{ik} = v_{ik} = 0$, not only do (1) and (37) simultaneously assume the semi-canonical form, but they are absolutely identical, so that *the surfaces S and S' are projective transformations of each other.*

Not every ruled surface of a linear congruence has the property in question. We have found that the necessary and sufficient conditions for this property are, that if (1) be written in the semi-canonical form ($p_{ik} = 0$), the coefficients q_{ik} become constants. Now suppose that the directrices of the congruence are distinct. Then since they are asymptotic lines of the surface, we may choose them as fundamental curves C_y and C_z. Our system (1) then assumes the form

$$y'' + q_{11} y = 0, \qquad z'' + q_{22} z = 0,$$

if the directrices be taken as opposite edges of the tetrahedron of reference. Moreover q_{11} and q_{22} are constants. Assuming the edge C_y of the tetrahedron to be the intersection of the planes $x_3 = 0$ and $x_4 = 0$, and the edge C_z the intersection of $x_1 = 0$, $x_2 = 0$, we find by elementary integration, if q_{11} and q_{22} are different from zero,

$$y_1 = e^{r_1 x}, \qquad y_2 = e^{r_2 x}, \qquad y_3 = 0, \qquad y_4 = 0,$$

$$z_1 = 0, \qquad z_2 = 0, \qquad z_3 = e^{r_3 x}, \qquad z_4 = e^{r_4 x},$$

where

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \pm \sqrt{-q_{11}}, \qquad \left.\begin{matrix} r_3 \\ r_4 \end{matrix}\right\} = \pm \sqrt{-q_{22}}.$$

If we put $\eta_k = ay_k + bz_k$, where a and b are constants, $\eta_1 \cdots \eta_4$ are the coördinates of a point on an arbitrary asymptotic line of the surface. This gives

$$\eta_1 = ae^{r_1 x}, \qquad \eta_2 = ae^{r_2 x}, \qquad \eta_3 = be^{r_3 x}, \qquad \eta_4 = be^{r_4 x}.$$

The relation between $\eta_1 \cdots \eta_4$ obtained by eliminating a, b from these equations, will be the equation of the ruled surface S. It is

$$\left(\frac{\eta_1}{\eta_2}\right)^{r_3 - r_4} \left(\frac{\eta_3}{\eta_4}\right)^{-(r_1 - r_2)} = 1,$$

or

$$\eta_1^{2\sqrt{-q_{22}}}\,\eta_4^{2\sqrt{-q_{11}}} - \eta_2^{2\sqrt{-q_{22}}}\,\eta_3^{2\sqrt{-q_{11}}} = 0.$$

But all such surfaces admit at least a two-parameter group of projective transformations.*

We have assumed q_{11} and q_{22} different from zero. If $q_{11} = q_{22} = 0$, S' is a ruled surface of the second order, admitting a six-parameter projective group.

Suppose $q_{11} = 0$, $q_{22} \neq 0$. Then we have

$$y_1 = x, \qquad y_2 = 1, \qquad y_3 = 0, \qquad y_4 = 0,$$

$$z_1 = 0, \qquad z_2 = 0, \qquad z_3 = e^{r_3 x}, \qquad z_4 = e^{r_4 x}.$$

The equation of the surface becomes

$$\frac{\eta_3}{\eta_4} = e^{2\sqrt{-q_{22}}\frac{\eta_1}{\eta_2}},$$

and this admits a three parameter group of projective transformations.

* LIE, *Theorie der Transformationsgruppen*, vol. III, p. 198.

Suppose now that the directrices of the congruence coincide. Taking the directrix as C_y and an asymptotic line for C_z, we may assume the system to have the form

$$y'' = 0, \qquad z'' = ky,$$

where k is a constant. (*Geometry*, p. 358.) A fundamental system of solutions is given by

$$y_1 = x, \qquad y_2 = 1, \qquad y_3 = 0, \qquad y_4 = 0.$$
$$z_1 = \tfrac{1}{6}kx^3, \qquad z_2 = \tfrac{1}{2}kx^2, \qquad z_3 = x, \qquad z_4 = 1.$$

The equation of S is found to be

$$\eta_1\eta_4^2 - \eta_2\eta_3\eta_4 + \tfrac{1}{3}k\eta_3^3 = 0,$$

which by the transformation

$$\eta_1 = x_3, \qquad \eta_2 = 3\sqrt[3]{\frac{k}{6}}\,x_2, \qquad \eta_3 = \sqrt[3]{\frac{6}{k}}\,x_1, \qquad \eta_4 = x_4,$$

becomes

$$x_3x_4^2 - 3x_1x_2x_4 + 2x_1^3 = 0,$$

or CAYLEY'S cubic scroll, which admits a three-parameter group of projective transformations.

We have proved the following theorem. *If the curves on the surface S', corresponding to the asymptotic lines on S, are also asymptotic lines, then S must be contained in a linear congruence, and must admit at least a two-parameter group of projective transformations.*

We conclude the consideration of the covariant P by showing that it cannot vanish identically. For, suppose P were zero for all values of x. Then would

$$\rho_k = \lambda y_k, \qquad \sigma_k = \lambda z_k,$$

i. e., the surfaces S and S' would coincide generator for generator. Moreover, if we introduce the expressions for ρ_k and σ_k, we would have,

$$2y_k' + (p_{11} - \lambda)y_k = -p_{12}z_k,$$
$$2z_k' + (p_{22} - \lambda)z_k = -p_{21}y_k,$$

i. e., the generator of the surface would be tangent to both C_y and C_z at every pair of corresponding points, which is manifestly impossible unless the surface degenerates into a straight line.

The singular generators which are tangent to C_y and C_z simultaneously are the ones for which P can vanish.

§ 4. *The covariant C.*

Let us put

$$\eta = \tfrac{1}{2}uy + u_{12}z,$$

(39)

$$\zeta = u_{21}y - \tfrac{1}{2}uz,$$

where

(40) $$u = u_{11} - u_{22}.$$

Then

(41) $$C = \eta z - \zeta y.$$

If in (39) we put $y = y_k$, $z = z_k$, we have the coördinates of two new points on the line L_{yz}, dividing the segment $P_y P_z$ in the ratios $2u_{12} : u$ and $-u : 2u_{21}$ respectively. Denote them by P_η and P_ζ. These points are distinct if

$$\theta_4 \neq 0.$$

We may consider that (P_η, P_y) and (P_ζ, P_z) are two pairs of an involution on the line L_{yz}. Denote by λ and λ' the ratios in which any point of the line L_{yz}, and that corresponding to it in the involution divide the segment $P_y P_z$. Then the parametric equation of the involution will be

(42) $$2u_{21}\lambda\lambda' + u(\lambda + \lambda') - 2u_{12} = 0.$$

The double points of the involution are determined by the quadratic equation obtained if in (42) λ' is put equal to λ. Therefore, they divide $P_y P_z$ in the ratios

(43) $$-\frac{u + \sqrt{\theta_4}}{2u_{21}} = \frac{2u_{12}}{u - \sqrt{\theta_4}} \quad \text{and} \quad -\frac{u - \sqrt{\theta_4}}{2u_{21}} = \frac{2u_{12}}{u + \sqrt{\theta_4}},$$

respectively. Of course, we may also find the double points by determining the pair of points dividing both $P_y P_\eta$ and $P_z P_\zeta$ harmonically.

If then we put

$$Y = \frac{u - \sqrt{\theta_4}}{2} y + u_{12}z = \rho\left[u_{21}y - \frac{u + \sqrt{\theta_4}}{2} z\right],$$

(44)

$$Z = \frac{u + \sqrt{\theta_4}}{2} y + u_{12}z = \sigma\left[u_{21}y - \frac{u - \sqrt{\theta_4}}{2} z\right],$$

where

(45) $$\rho = \frac{u - \sqrt{\theta_4}}{2u_{21}} = -\frac{2u_{12}}{u + \sqrt{\theta_4}}, \qquad \sigma = \frac{u + \sqrt{\theta_4}}{2u_{21}} = -\frac{2u_{12}}{u - \sqrt{\theta_4}},$$

Y_k and Z_k are the coördinates of the double points of the involution. In the case of real coefficients, it may therefore be classified as hyperbolic, parabolic or elliptic according as

$$\theta_4 \gtreqless 0.$$

But

(46) $$YZ = u_{12} C,$$

so that the double elements of the involution may be determined by factoring the covariant C.

Let us examine this involution more in detail. If $u_{12} = u_{21} = 0$, the parametric equation of the involution (42), reduces to $\lambda + \lambda' = 0$ provided that $u \neq 0$. Therefore, if $u_{12} = u_{21} = 0$ the involution is referred to its double points as fixed elements. It is worth while to show directly that $u_{12} = u_{21} = 0$ if the involution is referred to its double points. If y and z are transformed by the equations

(47) $$y = \alpha \bar{y} + \beta \bar{z}, \qquad z = \gamma \bar{y} + \delta \bar{z}, \qquad \alpha\delta - \beta\gamma = \Delta,$$

the quantities u_{ik} for the transformed system become $\bar{u}_{ik}$, where

(48) $$\begin{aligned} \Delta \bar{u}_{11} &= \alpha\delta u_{11} + \gamma\delta u_{12} - \alpha\beta u_{21} - \beta\gamma u_{22}, \\ \Delta \bar{u}_{12} &= \beta\delta u_{11} + \delta^2 u_{12} - \beta^2 u_{21} - \beta\delta u_{22}, \\ \Delta \bar{u}_{21} &= -\alpha\gamma u_{11} - \gamma^2 u_{12} + \alpha^2 u_{21} + \alpha\gamma u_{22}, \\ \Delta \bar{u}_{22} &= -\beta\gamma u_{11} - \gamma\delta u_{12} + \alpha\beta u_{21} + \alpha\delta u_{22}. \end{aligned}$$

This may be seen without much calculation by making use of the formulæ for the transformation of the quantities p'_{ik}, and also of the remark that the derivatives of $\alpha, \beta, \gamma, \delta$ cannot enter into equations (48), as is shown by the formulæ for the infinitesimal transformations of the quantities u_{ik}, which have been deduced on a former occasion (*Invariants*, equations (37)).

Now the double points of the involution are given by

$$Y = \frac{u - \sqrt{\theta_4}}{2} y + u_{12} z, \qquad Z = u_{21} y - \frac{u - \sqrt{\theta_4}}{2} z,$$

whence

$$\Delta y = -\frac{u - \sqrt{\theta_4}}{2} Y - u_{12} Z, \qquad \Delta z = -u_{21} Y + \frac{u - \sqrt{\theta_4}}{2} Z,$$

where

$$\Delta = \tfrac{1}{2} \sqrt{\theta_4} (u - \sqrt{\theta_4}).$$

If $\Delta \neq 0$, we can put in (48),

$$\alpha = \frac{u - \sqrt{\theta_4}}{2}, \qquad \beta = u_{12}, \qquad \gamma = u_{21}, \qquad \delta = -\frac{u - \sqrt{\theta_4}}{2},$$

which gives $\bar{u}_{12} = \bar{u}_{21} = 0$, as we wished to prove.

If $\Delta = 0$, we have either $\theta_4 = 0$, i. e., the double points of the involution coincide, or else

$$\theta_4 = u^2,$$

whence $u_{12} u_{21} = 0$, and therefore either u_{12} or u_{21} vanish in the first place, say $u_{12} = 0$, and P_y is already one of the double points. But this can always be avoided except in one case. For u_{12} cannot vanish identically for all curves on the surface unless $\bar{u}_{12} = 0$ follows from $u_{12} = 0$ for all values of the arbitrary functions α, β, γ, δ. But this is so if, and only if, $u_{11} - u_{22}$ and u_{21} also vanish, i. e., if, and only if, the surface is of the second order.

Therefore, *if the surface is not of the second order, the double points of the involution determined by* (39) *on every generator of the surface, are the intersections of that generator with the flecnode curve. In the case of a system with real coefficients this curve is real for those generators for which* $\theta_4 > 0$, *and imaginary for those generators for which* $\theta_4 < 0$. *If* θ_4 *vanishes identically, the flecnode curve intersects every generator in two coincident points. If* θ_4 *does not vanish identically, the values of* x *for which* $\theta_4 = 0$ *correspond to generators which intersect the curve in two coincident points, i. e., to generators which are either tangent to the curve or upon which there is a double point of the curve.*

The involution becomes indeterminate, as equations (39) show, if and only if $u = u_{12} = u_{21} = 0$, i. e., if the surface is of the second order. That this must be so is obvious geometrically. But even if u, u_{12}, u_{21} do not vanish identically, there will be in general particular values of x for which they do. As a matter of fact, if θ_4 does not vanish identically, let us take as our fundamental curves C_y and C_z the two branches of the flecnode curve. Then $u_{12} = u_{21} = 0$, and $u_{11} - u_{22} = u$ will not vanish identically, since $\theta_4 = u^2$. But unless θ_4 is a constant, there will always be values of x for which $u = 0$, so that there will then be values of x for which $u_{12} = u_{21} = u_{11} - u_{22} = 0$ simultaneously. Along such generators the involution is indeterminate. The osculating hyperboloid has more than three consecutive generators in common with S. We shall say that it *hyperosculates* the surface. Therefore, *every ruled surface, for which* θ_4 *is not a constant, has a certain number of generators along which the osculating hyperboloid hyperosculates the surface. The parameters of such generators are found from the equations*

$$u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

This singularity of ruled surfaces was mentioned to my knowledge for the first time by VOSS, * who also was the first to pay much attention to the flecnode curve.

* VOSS, *Zur Theorie der windschiefen Flächen*, Mathematische Annalen, vol. 8.

The flecnode curve may be an asymptotic line. But if it is, we have simultaneously

$$u_{12} = u_{21} = 0, \qquad p_{12} = p_{21} = 0,$$

whence follows also

$$q_{12} = q_{21} = 0,$$

so that in this case it consists of a pair of straight lines, which of course must not be coplanar.

Thus, if the flecnode curve is also an asymptotic line, it breaks up into a pair of distinct non-intersecting or coincident lines. The ruled surface then belongs to a linear congruence with distinct or coincident directrices, and these directrices are flecnode curves and asymptotic lines of the surface at the same time. If only one branch of the flecnode curve is an asymptotic line, then it is a straight line on the surface, and the surface belongs to a special linear complex.

The involution, which we have considered in this paragraph, has been considered before in the special case just mentioned, when the surface belongs to a linear congruence. According to a theorem first proved, I believe, by CREMONA,* we have an involution on every generator of such a surface determined as follows: *Every asymptotic line intersects every generator in two points. These points are the pairs of an involution whose double points are the intersections of the generator with the directrices of the congruence.*

This theorem enables us to give a concrete geometrical interpretation for the involution in the general case. Along a given generator g of the ruled surface S, construct the two flecnode tangents. Regard these as the directrices of a linear congruence. Consider any ruled surface, not of the second order, belonging to this congruence which has the generator g in common with S. Then every asymptotic line of this surface intersects g twice, and these two points of intersection constitute a pair of the involution.

We have seen how to transform the system of differential equation (1), so as to make $u_{12} = u_{21} = 0$. Equations (48) show that we may instead make $u_{11} - u_{22} = 0$. From (39) then follows at once, that the curves C_y and C_z are harmonic conjugates with respect to the two branches of the flecnode curve, if $u_{11} - u_{22} = 0$.

We have already noted that C vanishes identically, i. e., for absolutely arbitrary functions y and z, if and only if the integrating ruled surface is of the second order. But we can reduce the conditions under which this theorem holds. *If the covariant C vanishes for all possible curves C_y and C_z on the surface, the surface is of the second order.*

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*CREMONA, *Rappresentazione di una classe di superficie gobbe sopra un piano e determinazione delle loro curve assintotiche*, Annali di Mathematica, ser. 2, vol. 1 (1867-68).

For, suppose if possible that $C = 0$ for all curves C_y and C_z on S, and that $\theta_4 \neq 0$. Then the flecnode curve has two distinct branches, and, taking them as the fundamental curves, we shall have $u_{12} = u_{21} = 0$, whence since $C = 0$ follows $u_{11} - u_{22} = 0$, i. e., $\theta_4 = 0$. Therefore θ_4 cannot be different from zero. Let $\theta_4 = 0$ together with $C = 0$. We can certainly assume $u_{12} = 0$ by taking C_y to be the flecnode curve, unless the surface is of the second order, in which case $u_{12} = u_{21} = u_{11} - u_{22} = 0$ for all curves on the surface. But if $u_{12} = 0$ and $\theta_4 = 0$, we find at once $u_{11} - u_{22} = 0$, and since $C = 0$ also $u_{21} = 0$. Therefore C vanishes, as was stated, if and only if the surface S is of the second order.

Of course, there is an involution of planes through every generator of a ruled surface corresponding to the involution of points considered in this paragraph.

§ 5. *On the validity of certain transformations.*

In a former paper (*Geometry*, p. 347), it has been noted that the values of x corresponding to cuspidal generators are those, for which

$$D = \begin{vmatrix} y_1', & z_1', & y_1, & z_1 \\ y_2', & z_2', & y_2, & z_2 \\ y_3', & z_3', & y_3, & z_3 \\ y_4', & z_4', & y_4, & z_4 \end{vmatrix} = 0.$$

This however should be taken with a proviso. For, since $y_1 \cdots y_4$ have been interpreted as the homogeneous coördinates of a point, it is not admissible that they should vanish or become infinite simultaneously. For, in such cases, the point would be indeterminate. We must therefore assume that $y_1 \cdots y_4$, and similarly $z_1 \cdots z_4$ do not become zero or infinite simultaneously. This condition may always be fulfilled by multiplying y and z by properly chosen functions of x. Let us assume that this condition is fulfilled, and let $D = 0$ for $x = a$. Then there is a pinchpoint on the generator corresponding to $x = a$.

We can furnish a new proof of the fact that D must vanish for the values of x corresponding to cuspidal generators of the surface.

As in *Reciprocal Systems*, equations (5), let us write

$$D = \sum_{k=1}^{4} y_k' v_k = -\sum_{k=1}^{4} z_k' u_k.$$

Then u_k and v_k are the coördinates of the planes tangent to S at P_y and P_z respectively. If P_y and P_z are transformed into $P_{\bar{y}}$ and $P_{\bar{z}}$ by the equations

$$\bar{y} = \alpha y + \beta z, \qquad \bar{z} = \gamma y + \delta z,$$

these planes are transformed into the planes tangent to S at $P_{\bar{y}}$ and $P_{\bar{z}}$ by the equations

$$\bar{u} = \Delta(\alpha u + \beta v), \qquad \bar{v} = \Delta(\gamma u + \delta v),$$

where

$$\Delta = \alpha\delta - \beta\gamma.$$

Therefore if $v_k = \rho u_k$, i. e., if the tangent planes at P_y and P_z coincide, $\bar{u}_k$ and $\bar{v}_k$ are also proportional to u_k, i. e., *if the tangent planes at two points of a generator of the ruled surface coincide, it must be a cuspidal generator*, for the surface then has the same tangent plane at all points of the generator.

But then

$$D = \sum_{k=1}^{4} y'_k v_k = \rho \sum_{k=1}^{4} y'_k u_k = 0,$$

since u_k is the minor of z'_k in the determinant D.

We have shown (*Geometry*, equation (11)), that

(49) $$\frac{1}{D}\frac{dD}{dx} = -(p_{11} + p_{22}),$$

whence, if $D \neq 0$,

(50) $$D = Ce^{-\int (p_{11} + p_{22})dx},$$

where C is a constant different from zero. Since by a transformation

(51) $$y = \eta e^{-\frac{1}{2}\int p_{11}dx}, \qquad z = \zeta e^{-\frac{1}{2}\int p_{22}dx},$$

we can always make p_{11} and p_{22} vanish, we can make D a constant, and thus apparently lose sight of the pinchpoints. But clearly such a transformation (51) is strictly legitimate only for regions of the surface which contain no cuspidal generators. For, suppose that $x = a$ corresponds to a cuspidal generator, and neither $(y_1 \cdots y_4)$ nor $(z_1 \cdots z_4)$ vanish or become infinite simultaneously for $x = a$. Then either the system $(\eta_1 \cdots \eta_4)$ or $(\zeta_1 \cdots \zeta_4)$ or both will become simultaneously zero or infinite for $x = a$.

Similar remarks may be made about a transformation of the invariant of weight four, θ_4. This vanishes for those values of x which correspond to generators, whose two intersections with the flecnode curve coincide. If a transformation of the independent variable be made

$$\xi = \xi(x),$$

we have

$$\xi'(x)^4 \theta_4(\xi) = \theta_4(x).$$

Therefore, if in an interval in which θ_4 does not vanish, we put

$$\xi = \int \sqrt[4]{\theta_4(x)}\, dx,$$

we shall have

$$\theta_4(\xi) = 1,$$

but clearly this is not legitimate for values of x for which $\theta_4 = 0$.

A similar transformation may be applied to any invariant which does not vanish identically.

§ 6. *The covariant* C_3.

We may write the semi-covariants E and N of § 2 as follows:

$$(52)\quad E = \begin{vmatrix} \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, & y \\ v_{21}y - \frac{1}{2}(v_{11} - v_{22})z, & z \end{vmatrix}, \qquad N = \begin{vmatrix} (u_{11} - u_{22})\rho + 2u_{12}\sigma, & y \\ 2u_{21}\rho - (u_{11} - u_{22})\sigma, & z \end{vmatrix}.$$

Therefore

$$(53)\quad C_3 = E + 2N = \begin{vmatrix} 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, & y \\ 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \frac{1}{2}(v_{11} - v_{22})z, & z \end{vmatrix}.$$

If we take as curves C_y and C_z the two branches (assumed distinct), of the flecnode curve, we shall have $u_{12} = u_{21} = 0$, and therefore more simply

$$(54)\quad C_3 = \begin{vmatrix} 2(u_{11} - u_{22})\rho + (u'_{11} - u'_{22})y - p_{12}(u_{11} - u_{22})z, & y \\ -2(u_{11} - u_{22})\sigma + p_{21}(u_{11} - u_{22})y - (u'_{11} - u'_{22})z, & z \end{vmatrix},$$

for (*Invariants*, equations (32)), if $u_{12} = u_{21} = 0$,

$$(55)\quad v_{11} - v_{22} = 2(u'_{11} - u'_{22}),\quad v_{12} = -p_{12}(u_{11} - u_{22}),\quad v_{21} = +p_{21}(u_{11} - u_{22}).$$

Moreover θ_4 reduces to $(u_{11} - u_{22})^2$. If, then, we put

$$\xi = \int \sqrt{u_{11} - u_{22}}\, dx,$$

and if we introduce ξ as the independent variable in place of x, $u_{11} - u_{22}$ will be a constant, and, therefore,

$$u'_{11} - u'_{22} = 0.$$

These transformations are legitimate simultaneously for a region of the surface, which includes in its interior no generator along which the osculating hyperboloid hyperosculates the surface.

C_3 is thus simplified to

$$C_3 = (u_{11} - u_{22}) \begin{vmatrix} 2\rho - p_{12}z, & y \\ -2\sigma + p_{21}y, & z \end{vmatrix}. \tag{56}$$

Let us put

$$\alpha = 2\rho - p_{12}z, \qquad \beta = -2\sigma + p_{21}y, \tag{57}$$

and interpret these quantities geometrically.

The lines P_yP_z, $P_\rho P_\sigma$, P_yP_ρ and P_zP_σ form a skew quadilateral. P_yP_z is a generator of S, P_y and P_z being its intersection with the flecnode curve of S. $P_\rho P_\sigma$ is the generator of S' corresponding to P_yP_z, so that P_yP_z and $P_\rho P_\sigma$ are generators of the first kind and P_yP_ρ and P_zP_σ generators of the second kind of the hyperboloid osculating S along P_yP_z. Complete the tetrahedron determined by these four points by joining P_yP_σ and P_zP_ρ. Then α_k and β_k are clearly the coördinates of two points P_α and P_β on P_zP_ρ and P_yP_σ respectively. *These points are therefore reciprocal poles of each other with respect to the hyperboloid osculating S along P_yP_z.*

From the equations defining ρ and σ, we have

$$2y' = \rho - p_{11}y - p_{12}z,$$
$$2z' = \sigma - p_{21}y - p_{22}z,$$

and these are obviously the coördinates of two points $P_{y'}$ and $P_{z'}$ on the tangents to C_y and C_z at P_y and P_z respectively. Moreover, since $u_{12} = u_{21} = 0$, they are points on the two tangents to the flecnode curve. We may assume $p_{11} = p_{22} = 0$, for transformation (51) will make p_{11} and p_{22} vanish, and such a transformation clearly does not affect the geometrical significance of any of the quantities involved.

Then the points $P_{y'}$ and $P_{z'}$ determined by

$$\begin{aligned} 2y' &= \rho - p_{12}z, \\ 2z' &= \sigma - p_{21}y, \end{aligned} \tag{58}$$

are clearly the intersections of the tangents of the flecnode curve with P_zP_ρ and P_yP_σ respectively.

But from these equations we see that we may write

$$2y' = \tfrac{1}{2}(\alpha - p_{12}z), \qquad \rho = \tfrac{1}{2}(\alpha + p_{12}z),$$
$$2z' = -\tfrac{1}{2}(\beta + p_{21}y), \qquad \sigma = -\tfrac{1}{2}(\beta - p_{21}y).$$

so that the points $P_{y'}$ and P_ρ are harmonic conjugates with respect to P_α and P_z on the line $P_z P_\rho$, and similarly the points $P_{z'}$, P_σ, P_β and P_y form a harmonic range on $P_y P_\sigma$.

We thus have, besides S', a third ruled surface S'' associated with S in the following manner: Let P_y and P_z be the two flecnodes, supposed distinct, on a given generator of S. At each of these points three important lines intersect, viz., the generator, the flecnode tangent, and the tangent to the flecnode curve. All of these are in the plane tangent to the surface S at that point. In this plane pencil we construct a fourth line, so that with respect to it and the generator, the other two lines shall be harmonic conjugates. This line intersects the line joining the other flecnode to P_ρ in the point P_α. The point P_β is constructed in a similar way from a pencil having its vertex at P_z. The line $P_\alpha P_\beta$ is the generator of the surface S'' which corresponds to the generators $P_y P_z$ of S and $P_\rho P_\sigma$ of S'.

If, instead of the particular surface S', another of the covariant surfaces of § 3 had been chosen, by considering

$$\rho + \lambda y, \qquad \sigma + \lambda z$$

instead of ρ and σ, P_α and P_β would merely be displaced along the lines $P_\alpha P_y$ and $P_\beta P_z$ respectively. We should always obtain three ruled surfaces closely associated with each other.

The covariant C_3 vanishes identically if the ruled surface is of the second order. If it were to vanish in any other case we should have

$$2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z = \lambda y,$$
$$4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z = \lambda z,$$

i. e., the four points P_y, P_z, P_ρ, P_σ would be coplanar or, in subcases, even collinear. In other words the osculating hyperboloid would be degenerate, and the surface would be developable. *Therefore, C_3 can vanish identically only if the surface is of the second order.*

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